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ON THE NORMAL CONFIGURATION OF CONSERVATIVE SYSTEMS

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We consider the problem of reducing conservative systems to normal coordinates by means of the method of regularizing time transformation widely used in celestial mechanics. Because it is well known that canonic variables have equal validity we can consider the reduction of the systems also to normal momenta. In connection with the reduction mentioned we introduce the concepts of normal and incompletely normal system configurations. We study the existence conditions for normal configurations, proceeding from the structural properties of the Hamiltonian. In particular, we examine these conditions for systems with complete connections, systems with two degrees of freedom, Liouville-type systems, homogeneous systems, systems admitting of a similarity transformation group, systems possessing radial symmetry, and some others.

1. Definitions and Statement of the Problem. We consider a conservative system with k degrees of freedom, moving in a certain force field with energy constant h. The Hamilton-Jacobi equation has the form

$$H\left(q_1, q_2, \ldots, q_k, \quad \partial W/\partial q_1, \ldots, \partial W/\partial q_k\right) = h \tag{1.1}$$

We say that the system is reduced to normal coordinates and forms a normal configuration A $(q_1, q_2, \ldots, q_k)$ relative to the generalized coordinates q_j, if for a suitable choice of coordinates q_j and reduction time τ the equations of motion split up into k differential equations of the form

$$q_j'' = \sigma_j q_j \qquad (\sigma_j = \text{const}, \ j = 1, 2, \ldots, k) \tag{1.2}$$

Here τ is a suitably chosen regularizing variable depending, in the general case, on time t ; a prime indicates the derivative with respect to τ. A simple example of a normal configuration without the regularization of time t is the reduction to normal coordinates in the case of small oscillations of a system close to the equilibrium position [1].

Because it is well known that the canonical variables p_j and q_j have equal validity, we can analogously talk about normal configurations of systems also with respect to momenta p_j, if only the p_j satisfy differential equations of the form

$$p_j'' = \gamma_j p_j \qquad (\gamma_j = \text{const}, \ j = 1, 2, \ldots, k) \tag{1.3}$$

A system possesses an incomplete or partial normal configuration if Eqs. (1.2) hold not for all the indices $j = 1, 2, \ldots, k$, but for only a part of them, namely, $j = 1, 2, \ldots, \nu$ $(1 \leqslant \nu < k)$. In analogous fashion we can speak of an incomplete normal configuration of the system with respect to the momenta p_j also.

It may happen that some of the σ_j are equal to each other, then we have the case of degeneracy [2, 3]. In particular, the case of complete degeneracy obtains when all the σ_j are equal to each other: $\sigma_j = \sigma$ $(j = 1, 2, \ldots, k)$. Certain celestial mechanics problems may serve as examples. Thus, for example, in the n-body problem in the special case of the so-called constant configurations there obtains a normal configuration corresponding to the case of complete degeneracy [4], namely, when the centrobaric bodies perform Keplerian motions (with a period common to all of them) along circles with centers situated at the system's center of mass. The so-called central configurations in the n-body problem [5, 6] may serve as another example.

The problem to be considered below consists in the determination of existence conditions for normal configurations, as a function of the structural properties of the Hamiltonian $H = H(\mathbf{p}, \mathbf{q})$.

2. Fundamental Dependencies. The state of conservative system (1.1) is described by the canonic system of Hamiltonian equations

$$dq_j / dt = \partial H / \partial p_j, \qquad dp_j / dt = -\partial H / \partial q_j \qquad (j = 1, 2, \ldots, k) \tag{2.1}$$

Differentiating (2.1) and introducing the Poisson brackets

$$(\varphi, \psi) = \sum_{i=1}^{k} \left(\frac{\partial \varphi}{\partial q_i} \frac{\partial \psi}{\partial p_i} - \frac{\partial \varphi}{\partial p_i} \frac{\partial \psi}{\partial q_i} \right) \tag{2.2}$$

we obtain

$$\ddot{q}_j = (\dot{q}_j, H), \qquad \ddot{p}_j = (\dot{p}_j, H) \tag{2.3}$$

Here the dot denotes the derivative with respect to t. Hence by virtue of (1.2), (2.1) and (2.2) we can obtain a condition which the Hamiltonian function H should satisfy in case there exists a normal configuration without a regularizing time, i.e., when $\tau = t$

$$\sum_{i=1}^{k}\left(\frac{\partial^2 H}{\partial q_i \partial p_j}\frac{\partial H}{\partial p_i} - \frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial H}{\partial q_i}\right) = \sigma_j q_j \tag{2.4}$$

In the case when the function H $(\mathbf{p}, \mathbf{q})$ is a superposition of functions each of which depends only on one pair of canonic variables p_j and q_j ,

$$H(\mathbf{p}, \mathbf{q}) = H_1(p_1, q_1) + H_2(p_2, q_2) + \ldots + H_k(p_k, q_k) \tag{2.5}$$

condition (2.4) takes the simpler form

$$\frac{\partial^2 H_j}{\partial q_j \partial p_j}\frac{\partial H_j}{\partial p_j} - \frac{\partial^2 H_j}{\partial p_j^2}\frac{\partial H_j}{\partial q_j} = \sigma_j q_j \qquad (j = 1, 2, \ldots, k) \tag{2.6}$$

If we make use of the integrals

$$H_j(p_j, q_j) = \alpha_j \qquad (\alpha_1 + \alpha_2 + \ldots + \alpha_k = h) \tag{2.7}$$

and note that

$$\frac{\partial H_j}{\partial q_j} + \frac{\partial H_j}{\partial p_j}\frac{\partial p_j}{\partial q_j} = 0$$

then after some simplification and a subsequent integration condition (2.6) can be reduced to the form

$$\left(\frac{\partial H_j}{\partial p_j}\right)^2 = \sigma_j q_j^2 + c_j \qquad (c_j = \text{const}) \tag{2.8}$$

This result can be obtained in another way also if we make use of the first integral of the equations of motion (1.2).

T h e o r e m 2.1. Suppose that system (2.1) admits of a normal configuration both relative to the coordinates q_j as well as relative to the momenta p_j. Then there exist integrals of the form

$$\frac{\partial H}{\partial p_j} p_j + \frac{\partial H}{\partial q_j} q_j = \alpha_j \qquad (\alpha_j = \text{const}, \quad j = 1, 2, \ldots, k) \tag{2.9}$$

Indeed, let p_j and q_j satisfy the equations

$$q_j^{\cdot\cdot} = \sigma_j q_j, \qquad p_j^{\cdot\cdot} = \sigma_j p_j \qquad (j = 1, 2, \ldots, k)$$

Hence $q_j^{\cdot\cdot} p_j - p_j^{\cdot\cdot} q_j = 0$ and, consequently, as a result of integration we obtain $q_j^{\cdot} p_j - p_j^{\cdot} q_j = \alpha_j$ which by virtue of Eqs. (2.1) also leads to integrals of form (2.9). As an example, consider a natuıal conservative system with a Hamiltonian H $(\mathbf{p}, \mathbf{q})$ of the form

$$H = \frac{1}{2}\sum_{j=1}^{k}(p_j^2 - \sigma_j q_j^2) \qquad (H(\mathbf{p}, \mathbf{q}) = h)$$

Here the hypotheses of Theorem 2.1 are satisfied and there exist integrals of form (2.9) which may be written as

$$p_j^2 - \sigma_j q_j^2 = \alpha_j \qquad (\alpha_1 + \alpha_2 + \ldots + \alpha_k = 2h)$$

3. Time Regularization. In many cases we can succeed in detecting a normal configuration by means of introducing a new regularizing variable $\tau = \tau(t)$ defined by an integral of the form

$$\tau = \int^{t} \frac{dt}{u} \qquad (u = u(\mathbf{p}(t), \mathbf{q}(t)) \neq 0) \tag{3.1}$$

Here $u(\mathbf{p}, \mathbf{q})$ is a suitably selected continuous scalar function of the $2k$ variables p_j and q_j, not vanishing in that region G of the phase space E^{2k}, wherein the original system (2.1) is being considered.

Time transformations similar to (3.1) have been used, in particular, by Birkhoff [7] to study general Lagrangian systems and in other cases, for example when treating regularizing transformations in the three-body problem [7, 8]. Under the transformation (3.1) indicated Eqs. (2.1) lose their Hamiltonian form and take the form

$$\frac{dq_j}{d\tau} = u \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{d\tau} = -u \frac{\partial H}{\partial q_j} \qquad (j = 1, 2, \ldots, k) \tag{3.2}$$

However, as Poincaré [8] had noted, can once again be given a Hamiltonian form by introducing a new function $F(\mathbf{p}, \mathbf{q}) = u(H - h)$, where h is the energy constant for the solutions $p_j(t)$ and $q_j(t)$ of system (2.1). To be convinced of this it is sufficient to differentiate $F(\mathbf{p}, \mathbf{q})$ with respect to the variables p_j and q_j and to note that by virtue of the energy integral the equality $H(\mathbf{p}, \mathbf{q}) - h = 0$ is satisfied on the set of solutions $p_j(t)$ and $q_j(t)$ of system (2.1), corresponding to the energy constant h. System (3.2) takes the Hamiltonian form

$$\frac{dq_j}{d\tau} = \frac{\partial F}{\partial p_j}, \qquad \frac{dp_j}{d\tau} = -\frac{\partial F}{\partial q_j} \qquad (F(\mathbf{p}, \mathbf{q}) = h') \tag{3.3}$$

Here $F(\mathbf{p}, \mathbf{q})$ is the new Hamiltonian function and h' the new energy constant. It should be noted that with regard to transformation (3.1) not all solutions of system (3.3) will correspond to solutions of system (2.1) but only those which correspond to the value $h' = 0$ of the new energy constant. Solutions of system (3.3) for values $h' \neq 0$ of the energy constant have no relation whatsoever with the solutions of system (2.1).

Theorem 3.1. Suppose that as a result of transformation (3.1) system (2.1) admits of a normal configuration relative to the coordinates q_j and relative to the momenta p_j and, consequently,

$$q_j'' = \sigma_j q_j, \qquad p_j'' = \sigma_j p_j \qquad (j = 1, 2, \ldots, k) \tag{3.4}$$

Then there exist integrals of the form

$$u(\mathbf{p}, \mathbf{q})(q_j^{\cdot} p_j - p_j^{\cdot} q_j) = \alpha_j \qquad (\alpha_j = \text{const}) \tag{3.5}$$

For the proof we set up the difference between Eqs. (3.4) having first multiplied them by p_j and q_j, respectively. By integrating we obtain $q_j' p_j - p_j' q_j = \alpha_j$. We can give form (3.5) to these integrals if we only note that by virtue of (3.1), $dt = u d\tau$. We assert that a theorem analogous to 3.1 can be formulated also in the case of an incomplete normal configuration.

4. Systems with Complete Connections [9]. Since a natural conservative system with one degree of freedom is invertible [5], its Lagrangian L can be written as follows:

$$L = {}^1/_2\, m(q) q^{\cdot 2} - V(q) \tag{4.1}$$

Here $V(q)$ is the potential energy and $m(q)$ is a positive function, the so-called equivalent mass of the system [10].

Theorem 4.1. Let a natural conservative system with one degree of freedom (4.1) move in a force field with potential $V = {}^1/_2\sigma q^2$. Then this system admits of a normal configuration.

Indeed, having written out the energy integral and by introducing a new variable and setting $d\tau = dt / (m(q))^{1/2}$, we obtain $q'' = \sigma q$.

Theorem 4.2. Suppose that system (4.1) moves in force fields with potentials $V_1(q)$ and $V_2(q)$ of the form

$$V_1(q) = {}^1/_2\, \sigma m(q)\ (A - q^2), \quad V_2(q) = {}^1/_2\sigma\, (m(q))^{-1}\, (A - q^2) \qquad (4.2)$$

Then in both cases there exists a normal configuration for an energy constant $h = 0$.

To be convinced of this it is sufficient to write out the energy integral for the condition $h = 0$, to cancel, respectively, the nonzero factors $m(q)$ and $(m(q))^{-1}$ and to introduce in the case of $V_2(q)$ a new variable τ, by setting $d\tau = dt / m(q)$. As a result after differentiating we obtain $q^{\cdot\cdot} = \sigma q$, $q'' = \sigma q$ respectively. We see that in the case of $V_2(q)$ a normal configuration obtains also relative to the momentum p.

5. Systems with a Radial Symmetry. For such systems the Lagrange function L remains (for $k > 2$) invariant under an arbitrary rotation of the k-dimensional Euclidean space around the origin and has the form [5]

$$L = \frac{1}{2} g(r) \sum_{j=1}^{K} x_j^{\cdot 2} - V(r) \qquad (r^2 = x_1^2 + \ldots + x_k^2,\ g(r) > 0)$$

We introduce a new variable τ by setting $u = (g(r))^{1/2}$ and we set up the Lagrange equation

$$\frac{d^2 x_j}{d\tau^2} = -\frac{\partial V}{\partial r}\frac{x_j}{r} \qquad (dt = (g(r))^{1/2} d\tau)$$

Hence, in particular, it follows that

$$x_i x_j'' - x_j x''_i = (x_i x_j - x_j x_i)\frac{1}{r}\frac{\partial V}{\partial r}$$

and, consequently, there exist integrals of the form

$$x_i x_j' - x_j x_i' = \alpha_{ij} \qquad (\alpha_{ij} = \text{const})$$

Setting $V = -{}^1/_2\, \sigma r^2$ we obtain the normal configuration

$$x_j'' = \sigma x_j \qquad (j = 1, 2, \ldots, k)$$

6. Systems with Two Degrees of Freedom. For such systems the equations of motions in the uninvertible case can be given the normal form [5, 7]

$$x^{\cdot\cdot} - \Omega(x, y)\, y^{\cdot} = -V_x, \quad y^{\cdot\cdot} + \Omega(x, y)\, x^{\cdot} = -V_y \qquad (\Omega(x, y) \neq 0) \quad (6.1)$$

Certain problems of celestial mechanics [11] also lead to such equations.

Theorem 6.1. A conservative system with two degrees of freedom (6.1) in the uninvertible case ($\Omega \neq 0$) cannot be reduced to a normal configuration by means of time regularization (3.1).

Indeed, by introducing the new variable τ (3.1), where $u(x, y)$ is a properly selected function, and by using the transition formulas

$$\frac{d}{dt}=\frac{1}{u}\frac{d}{d\tau},\qquad \frac{d^2}{dt^2}=-\frac{1}{u^2}\frac{du}{dt}\frac{d}{d\tau}+\frac{1}{u^2}\frac{d^2}{d\tau^2}$$

we take (6.1) to the form

$$x''-u^{\cdot}x'-\Omega uy'=-u^2V_x,\qquad y''-u^{\cdot}y'+\Omega ux'=-u^2V_y \tag{6.2}$$

To obtain the normal configuration it is necessary to get rid of the terms which are linear with respect to the velocities, which leads to the two conditions

$$u^{\cdot}x''+\Omega uy'=0,\qquad \Omega ux'-u^{\cdot}y'=0$$

By equating the determinant of this system in the variables x' and y' to zero, for the determination of $u\ (x, y)$ we obtain the relation $u^{\cdot 2}+\Omega^2u^2=0$, which is impossible since $\Omega\neq 0$, and $u\ (x, y)$ is a real positive function $(v>0)$. The theorem is proved.

However, as regards systems of invertible type $(\Omega=0)$, here by means of time regularization (3.1) we can obtain a normal configuration. Thus, for example, in the two-body problem the system is reduced to a normal configuration by a Levi-Civita transformation [12].

7. Systems of Liouville Type. For such systems the Hamiltonian function $H\ (\mathbf{p},\mathbf{q})$ can be brought to the form

$$H=\frac{1}{2u(\mathbf{q})}\sum_{j=1}^{k}(p_j^2+2V_j(q_j))\qquad (H(\mathbf{p},\mathbf{q})=h) \tag{7.1}$$

$$u(\mathbf{q})=u_1(q_1)+u_2(q_2)+\ldots+u_k(q_k)\qquad (u(\mathbf{q})>0)$$

The Hamiltonian equations for Liouville-type systems (7.1) have the form

$$q_j^{\cdot}=\frac{1}{u}p_j,\qquad p^{\cdot}_j=\frac{1}{u}\frac{\partial}{\partial q_j}(hu_j(q_j)-V_j(q_j)) \tag{7.2}$$

Hence, by using a well-known method [1], we obtain the following system of equations:

$$uq_j^{\cdot}=(2\Phi_j(q_j))^{1/2}\qquad (\Phi_j(q_j)=hu_j(q_j)-V_j(q_j)+\gamma_j) \tag{7.3}$$

Here h is the energy constant and γ_j are the constants of integration satisfying, by virtue of the energy integral, the condition $\gamma_1+\gamma_2+\ldots+\gamma_k=0$.

Theorem 7.1. For the Liouville-type system (7.1) being considered let the functions $\Phi_j\ (q_j)$ in (7.3) have, for a specified value of the energy constant h the form: $\Phi_j\ (q_j)={}^1/_2\ \sigma_jq_j^2+c_j\ (c_j=\text{const})$. Then the Liouville system (7.1) admits of a normal configuration both relative to the coordinates q_j as well as relative to the momenta p_j.

To be convinced of this we introduce a new variable τ by setting $d\tau=dt/u\ (q)$ $(u\ (q)>0)$. We obtain

$$q'_j=(\sigma_jq_j^2+2c_j)^{1/2},\qquad q_j''={}^1/_2(\sigma_jq_j^2+2c_j)^{-1/2}2\sigma_jq_jq_j'=\sigma_jq_j$$

Further, we write down the following chain of equalities:

$$p_j=q_j',\qquad p_j'=q_j''=\sigma_jq_j,\qquad p_j''=\sigma_jq_j'=\sigma_jp_j$$

We remark that Theorem 7.1 is valid also in the case when $\Phi_j\ (q_j)$ is a quadratic function of the form $\Phi_j\ (q_j)={}^1/_2\sigma_jq_j^2+a_jq_j+b_j\ (a_j, b_j=\text{const})$, since this case can be reduced to the previous one by a linear substitution.

8. Homogeneous Systems. Such is the name given to Liouville-type systems (7.1) when $u(\mathbf{q}) \equiv 1$, while the kinetic energy $T(\mathbf{q}, \mathbf{q}^{\cdot})$ and the potential energy $V(\mathbf{q})$ are the superpositions of homogeneous functions:

$$T(\mathbf{q}, \mathbf{q}^{\cdot}) = \frac{1}{2}\sum_{j=1}^{k} a_j q_j^{\nu} q_j^{\cdot 2}, \qquad V(\mathbf{q}) = \sum_{j=1}^{k} c_j q_j^{n} \qquad (a_j, c_j = \mathrm{const}) \tag{8.1}$$

Having written out the equations of motion

$$a_j q_j^{\nu} q_j^{\cdot\cdot} + {}^1/_2 \nu a_j q_j^{\nu-1} q_j^{\cdot 2} + n c_j q_j^{n-1} = 0 \tag{8.2}$$

as a result of integrating we obtain a system of k first integrals

$${}^1/_2 a_j q_j^{\nu} q_j^{\cdot 2} + c_j q_j^{n} = h_j \qquad (h_1 + \ldots + h_k = h;\ j = 1, 2, \ldots, k) \tag{8.3}$$

where h is the energy constant for the original system (8.1).

Let us determine those values of parameters ν and n for which normal configurations obtain. Having made the change of variables $q_j = \xi_j^{-\alpha}$, where α is some constant, and having determined $q_j^{\cdot}$ and $q_j^{\cdot\cdot}$, we obtain the equations of motion in the following form:

$$-\alpha a_j \xi_j^{-\alpha\nu-\nu-1} \xi_j^{\cdot\cdot} + \alpha a_j \left(1 + \alpha + \frac{\alpha\nu}{2}\right) \xi_j^{\cdot 2} \xi_j^{-\alpha\nu-\nu-2} + n c_j \xi_j^{-n\alpha+\alpha} = 0$$

We now choose α in such a way that the coefficient of $\xi_j^{\cdot 2}$ vanishes. This yields $\alpha = -2(2+\nu)^{-1}$ $(2 + \nu \neq 0)$ and, consequently, as a result of simplifications we obtain

$$\xi_j^{\cdot\cdot} + \frac{1}{2} n (2 + \nu) \frac{c_j}{a_j} \xi_j^{\beta} = 0 \qquad \left(\beta = \frac{2n - \nu - 2}{2 + \nu}\right) \tag{8.4}$$

To obtain the normal configurations we must set $\beta = 1$. Here, $2 + \nu - n = 0$ and, consequently, (8.4) takes the form

$$\xi_j^{\cdot\cdot} + \sigma_j \xi_j = 0 \qquad (\sigma_j = {}^1/_2 n^2 c_j / a_j) \tag{8.5}$$

We assert that the relation $2 + \nu - n = 0$ obtained is the condition under which the natural Lagrangian systems (8.1) admit of a one-parameter group of geometric similarity transformations of the form $q_j^* = \lambda q_j$ [13].

Theorem 8.1. For homogeneous systems (8.1) let the degrees of homogeneity of the combined expression for the kinetic energy $T^*(\mathbf{q}, \mathbf{p})$ and the potential energy $V(\mathbf{q})$ relative to q_j be related by the condition $2 - \nu - n = 0$. Then there exist integrals of the form

$$(q_1^{\cdot} p_1 - p_1^{\cdot} q_1) + (q_2^{\cdot} p_2 - p_2^{\cdot} q_2) + \ldots + (q_k^{\cdot} p_k - p_k^{\cdot} q_k) = nh \tag{8.6}$$

Indeed, since the generalized momentum $p_j = a_j q^{\nu}{}_j q^{\cdot}{}_j$, the Hamiltonian function has the following form:

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\sum_{j=1}^{k} a_j^{-1} q_j^{-\nu} p_j^2 + \sum_{j=1}^{k} c_j q_j^{n} \tag{8.7}$$

Hence by virtue of Euler's theorem on homogeneous functions, and also from the energy integral $H(\mathbf{p}, \mathbf{q}) = h$, we obtain

$$\sum_{j=1}^{k} \left(\frac{\partial H}{\partial p_j} p_j + \frac{\partial H}{\partial q_j} q_j\right) = (2 - \nu - n) T^* + nh \tag{8.8}$$

which leads, by virtue of the condition $2 - \nu - n = 0$, to integrals (8.6).

Theorem 8.2. Suppose that a homogeneous system with Hamiltonian (8.7) admits of a normal configuration both relative to the coordinates q_j as well as relative to the momenta p_j. Then, the degrees of homogeneity of the combined expression for the kinetic energy T^* (q, p) and the potential energy V (q) relative to coordinates q_j are $\nu = 0$ and $n = 2$, respectively.

Indeed, by virtue of the theorem's hypotheses we have $q_j^{\cdot\cdot} = \sigma_j q_j$, $p_j^{\cdot\cdot} = \sigma_j p_j$ and, consequently, $q_j^{\cdot\cdot} p_j - p_j^{\cdot\cdot} q_j = 0$ which, after integration, yields $q_j^{\cdot} p_j - p_j^{\cdot} q_j = \alpha_j$ ($\alpha_j = \text{const}$). By summing the integrals obtained over index j and using the Hamiltonian equations (2.1), and also relations (8.8), we obtain

$$(2 - \nu - n)\ T^*(q, p) + nh = \alpha \qquad (\alpha_1 + \ldots + \alpha_k = \alpha)$$

which leads to the condition $2 - \nu - n = 0$. But for homogeneous systems (8.1) admitting of a normal configuration relative to coordinates q_j the condition $2 + \nu - n = 0$ must be fulfilled, as was shown above. Comparing the two conditions obtained involving ν and n, we obtain $\nu = 0$ and $n = 2$.

9. Similar Systems. Natural conservative systems (8.1) admitting of a group of similarity transformations

$$q_j' = \lambda q_j, \qquad t' = \tau t \qquad (\lambda, \tau = \text{const}) \tag{9.1}$$

fall in this class. This means that if a certain solution $q_j = q_j(t)$ of the system of equations exists, then a solution of the form

$$q_j' = \lambda q_j (\tau^{-1} t') = \lambda q_j (\lambda^{n/2 - \nu/2 - 1}\, t') \qquad (\lambda \neq 0) \tag{9.2}$$

exists also. This result follows from the fact that by virtue of transformations (9.1) the relation

$$\lambda^{2+\nu-n} \tau^{-2} = 1 \tag{9.3}$$

ensuing from similarity considerations [14] must be fulfilled for the systems (8.1) being examined.

Theorem 9.1. Let system (8.1) possess a normal configuration $q_j^{\cdot\cdot} = \sigma_j q_j$ ($j = 1, 2, \ldots, k$) and, furthermore, admit of the group of similarity transformations (9.1). Then the geometric similarity condition $2 + \nu - n = 0$ is fulfilled.

This result follows immediately from condition (9.3) as well as from the existence condition for a normal configuration, since here $\tau^{-2} = 1$. Note that for $\nu = 0$ together with the solutions $q_j = q_j(t)$ there exist, by virtue of (9.2), also the solutions

$$q_j' = \lambda q_i (\lambda^{n/2-1} t') \qquad (t' = t)$$

which coincides with the result obtained earlier by Wintner [5].

10. Generalized Systems. Conservative systems in which the Hamiltonian function H (p, q) has the form

$$H(\mathbf{p}, \mathbf{q}) = H^*(\mathbf{p}, \mathbf{q}) / u(\mathbf{p}, \mathbf{q}) \qquad (H(\mathbf{p}, \mathbf{q}) = h,\ u(\mathbf{p}, \mathbf{q}) > 0) \tag{10.1}$$

belong here. Here H^* and u are superpositions of functions H_j and u_j each of which depends only on p_j and q_j

$$H^* = H_1(p_1, q_1) + \ldots + H_k(p_k, q_k), \qquad u = u_1(p_1, q_1) + \ldots + u_k(p_k, q_k) \tag{10.2}$$

These systems are a generalization of Liouville-type systems and fall into the class of integrable systems [15]. By virtue of (10.1) and (10.2) and of the energy integral $H^*(\mathbf{p},\ \mathbf{q})-hu\ (\mathbf{p},\ \mathbf{q})=0$, the Hamiltonian equations, after some manipulations, can be brought to the form

$$u\frac{dq_j}{dt}=\frac{\partial K_j}{\partial p_j},\qquad u\frac{dp_j}{dt}=-\frac{\partial K_j}{\partial q_j}\qquad (K_j(p_j,q_j)=H_j-hu_j)\qquad (10.3)$$

Theorem 10.1. Let $H_j\ (p_j,\ q_j)$ and $u_j\ (p_j,\ q_j)$ be homogeneous functions of degrees m and n, respectively, in each of the variables p_j and q_j. Then the bilinear form $\Omega\ (\mathbf{p},\ \mathbf{q})=p_1q_1+p_2q_2+\ldots+p_kq_k$ of the canonical variables p_j and q_j is an integral of the equations of motion.

Indeed, by multiplying Eqs. (10.3) by p_j and q_j, respectively, and adding them, after summing over index j we obtain

$$u\frac{d\Omega}{dt}=\sum_{j=1}^{k}\left(\frac{\partial H_j}{\partial p_j}p_j-\frac{\partial H_j}{\partial q_j}q_j\right)-h\sum_{j=1}^{k}\left(\frac{\partial u_j}{\partial p_j}p_j-\frac{\partial u_j}{\partial q_j}q_j\right)\qquad (10.4)$$

Hence, by virtue of the theorem's hypotheses and also of Euler's theorem on homogeneous functions, we obtain $\Omega\ (\mathbf{p},\ \mathbf{q})=\mathrm{const}$.

Theorem 10.2. Let the hypotheses of Theorem 10.1 regarding the degrees of homogeneity of the functions $H_j\ (p_j,\ q_j)$ and $u_j\ (p_j,\ q_j)$ be fulfilled. Then there exists an integral of the form

$$(q_1p_1-p_1^{\cdot}q_1)+\ldots+(q_k^{\cdot}p_k-p_k^{\cdot}q_k)=2h\,(m-n)\qquad (10.5)$$

In order to be convinced of this we multiply Eqs. (10.3) by p_j and q_j, respectively, and we take their difference. After summing over index j we obtain, by virtue of Euler's theorem on homogeneous functions,

$$u\,(\mathbf{p},\ \mathbf{q})\,((q_1^{\cdot}p_1-p_1^{\cdot}q_1)+\ldots+(q_k^{\cdot}p_k-p_k^{\cdot}q_k))=2mH^*(\mathbf{p},\ \mathbf{q})-2nhu\,(\mathbf{p},\ \mathbf{q})$$

Replacing here $H^*\ (\mathbf{p},\ \mathbf{q})$ by $hu\ (\mathbf{p},\ \mathbf{q})$, which follows directly from the energy integral, we obtain (10.5) after cancelling the nonzero factor $u\ (\mathbf{p},\ \mathbf{q})$.

Theorem 10.3. Let H depend only on p_j and be a homogeneous function of degree m of p_j and let u_j depend only on q_j and be a homogeneous function of degree n of q_j. Then for nonzero values of the energy constant h and for $m+n\neq 0$ there exists an integral admitting of a decomposition into a bilinear form $\Omega\ (\mathbf{p},\ \mathbf{q})$ and a secular term $(m+n)ht$.

Indeed, by virtue of Euler's theorem on homogeneous functions, from (10.4) we obtain $u\dot{\Omega}=mH^*+hnu=u\,(m+n)h$. Cancelling the nonzero factor $u\ (\mathbf{p},\mathbf{q})$ and integrating, we obtain the integral

$$\Omega\ (\mathbf{p},\ \mathbf{q})=(m+n)ht+\mathrm{const}\qquad (10.6)$$

Corollary 10.3. When the hypotheses in the preceding theorem, regarding the degrees of homogeneity, are fulfilled, the bilinear form $\Omega\ (\mathbf{p},\ \mathbf{q})$ is an integral of the equations of motion in the two cases: a) $h=0$, b) $(m+n)=0$. This follows directly from integral (10.6).

Theorem 10.4. Let $H_j\ (p_j,\ q_j)$ and $u_j\ (p_j,\ q_j)$ be quadratic functions of the form

$$H_j={}^1/_2\,(a_jp_j^2+c_jq_j^2),\quad u_j={}^1/_2\,(b_jq_j^2+d_jp_j^2)\qquad (a_j,\ b_j,\ c_j,\ d_j=\mathrm{const})\qquad (10.7)$$

Then system (10.1) admits of a normal configuration both relative to the coordinates q_j as well as relative to the momenta p_j.

Indeed, by introducing a new variable τ by setting $d\tau = dt/u$, by virtue of (10.7) Eqs. (10.3) reduce to the form

$$q_j' = (a_j - hd_j)\,p_j, \qquad p_j' = (-c_j + hb_j)\,q_j$$

Hence there follows immediately the existence of normal configurations both relative to the coordinates as well as relative to the momenta,

$$q_j'' = \sigma_j q_j, \qquad p_j'' = \sigma_j p_j \qquad (\sigma_j = (a_j - hd_j)(-c_j + hb_j))$$

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